Chapter 1

Introduction

Contents

1.1 Motivation 1
1.2 Basics 1
1.3 Preliminary algebra 3
1.4 Coordinate geometry 4
1.5 Partial fractions 5
1.6 Binomial expansion 5
1.7 The principle of mathematical induction (MI or mi) . . . 6

1.1 Motivation

Why do we study calculus?
Calculus is an important branch of mathematics and is concerned with two basic operations called **differentiation** and **integration.** These operations are related and both rely for their definitions on the use of **limits.**
We study calculus with some revision of A-level work. We introduce the basic concepts of mathematical analysis.

1.2 Basics

Notation: We use the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ for the set of natural numbers, integers and real numbers, respectively:

$$\mathbb{N} = \{0, 1, 2, 3, \dots\} \qquad \mathbb{Z} = \{0, 1, 2, 3, \dots\} \cup \{-1, -2, -3, -4 \dots\}$$

Given two real numbers $a < b$ we use the following interval notations:

$$[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\},$$

$$[a, b) = \{x \in \mathbb{R} | a \leq x < b\},$$

$$(a, b] = \{x \in \mathbb{R} | a < x \leq b\},$$

$$(a, b) = \{x \in \mathbb{R} | a < x < b\}.$$

We also use the notations

$$[a, +\infty) = \{x \in \mathbb{R} | a \leq x\},$$

$$(a, +\infty) = \{x \in \mathbb{R} | a < x\},$$

$$(-\infty, a] = \{x \in \mathbb{R} | x \leq a\},$$

$$(-\infty, a) = \{x \in \mathbb{R} | x < a\}.$$

Functions: We shall use notations like

$$f : [a, b] \longrightarrow \mathbb{R}$$

to convey that f is a real-valued function defined for all $a \leq x \leq b$, *i.e.,* $f(x)$ is a real number for all $x \in [a, b]$. In general, $g : A \longrightarrow B$ conveys that g is a function with **domain** A and **range** B, *i.e.,* for all $x \in A$, $g(x) \in B$. $g(x)$ is the **value of** g **at** x**.**

Definition (provisional): A function is a rule which assigns, to each real number in its domain, some real number in its range.

1.2.1 Examples of functions

1. $f_1 : \mathbb{R} \longrightarrow \mathbb{R}, f_1(x) = x$
2. $f_2 : [0, +\infty) \longrightarrow [0, +\infty), f_2(x) = \sqrt{x}$
3. $f_3 : (0, +\infty) \longrightarrow (0, +\infty), f_3(x) = \frac{1}{x}$
4. $$f_4 : \mathbb{R} \longrightarrow \mathbb{R}, f_4(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 2 & \text{if } x \text{ is irrational} \end{cases}$$
5. $$f_5 : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}, f_5(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$$

Building new functions: Suppose

$$f, g : A \longrightarrow \mathbb{R}.$$

Define

$$f + g : A \longrightarrow \mathbb{R}$$

by $(f+g)(x) = f(x) + g(x)$, the **sum** of f and g.

$$f - g : A \longrightarrow \mathbb{R}$$

by $(f-g)(x) = f(x) - g(x)$, the **difference** of f and g.

$$(f \cdot g) : A \longrightarrow \mathbb{R}$$

by $(f \cdot g)(x) = f(x) \cdot g(x)$, the **product** of f and g.
If $g(x) \neq 0$ for all $x \in A$, we also define

$$\left(\frac{f}{g}\right) : A \longrightarrow \mathbb{R}$$

by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, the **quotient** of f and g.

1.2.2 Examples

$$f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = x^2$$
$$g : \mathbb{R} \longrightarrow \mathbb{R}, g(x) = \frac{1}{1+x^2}$$

$(f \cdot g)(x) = f(x) \cdot g(x) = x^2 \cdot \frac{1}{1+x^2} = \frac{x^2}{1+x^2}$, $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2}{\frac{1}{1+x^2}} = x^2(1+x^2) = x^4 + x^2$.

1.3 Preliminary algebra

1.3.1 Polynomial functions

Definition: A polynomial of degree n is a function $f(x)$ of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0.$$

Here n is an integer $n > 0$, called the degree of the polynomial f.

Definition: A polynomial equation is an equation of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0.$$

It is satisfied by particular values of x, called the roots of the polynomial.
For $n = 1$, (linear case) we have

$$a_1 x + a_0 = 0 \Rightarrow x = -\frac{a_0}{a_1}.$$

For $n = 2$, (quadratic case) we have

$$a_2 x^2 + a_1 x + a_0 = 0 \Rightarrow x_\pm = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}.$$

Theorem 1.1: An n-th degree polynomial equation has exactly n roots.

1.3.2 Factorising polynomials

We have just seen that a polynomial equation can be written in any of the following alternative forms

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0, \\ f(x) &= a_n (x-\alpha_1)^{m_1} (x-\alpha_2)^{m_2} \cdots (x-\alpha_r)^{m_r}, \\ f(x) &= a_n (x-\alpha_1)(x-\alpha_2) \cdots (x-\alpha_n), \end{aligned}$$

with $m_1 + m_2 + \cdots + m_r = n$.

Example 1.1: The roots of a quadratic polynomial $f(x) = a_2 x^2 + a_1 x + a_0 = 0$ are α_1 and α_2, such that

$$\alpha_1 + \alpha_2 = -\frac{a_1}{a_2} \qquad \alpha_1 \cdot \alpha_2 = \frac{a_0}{a_2}.$$

1.3.3 Trigonometric identities

Single-angle identities:

$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1, \\ 1 + \tan^2\theta &= \sec^2\theta, \\ \cot^2\theta + 1 &= \csc^2\theta. \end{aligned}$$

Compound-angle identities:

$$\begin{aligned} \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B, \\ \sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B. \end{aligned}$$

Double-angle identities:

$$\begin{aligned} \cos(2A) &= 1 - 2\sin^2 A, \\ \sin(2A) &= 2\sin A \cos A, \end{aligned}$$

1.4 Coordinate geometry

Equation of a straight-line: The standard form for a straight-line graph is

$$y = mx + c,$$

representing a linear relationship between the independent variable x and the dependent variable y. The slope m is equal to the tangent of the angle the line makes with the x-axis and c is the intercept of the y-axis. An alternative form for the equation of a straight line is

$$ax + by + k = 0,$$

with $m = -\frac{a}{b}$ and $c = -\frac{k}{b}$. This form treats x and y on a more symmetrical basis, the intercepts on the two axes being $-\frac{k}{a}$ and $-\frac{k}{b}$, respectively.

Equation of a line that passes through (x_1, y_1) and (x_2, y_2): Given two points (x_1, y_1) and (x_2, y_2), we find the equation of the line that passes through both of them as follows. The slope is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

and

$$y - y_1 = m(x - x_1)$$

or

$$y - y_2 = m(x - x_2)$$

is the desired linear relationship between y and x.

Example 1.2: Find the equation of the line that passes through the points $(1, 2)$ and $(5, 3)$.
`Do as an exercise at home.`

1.5 Partial fractions

Example 1.3: Express the function

$$f(x) = \frac{4x + 2}{x^2 + 3x + 2}$$

in partial fractions.
We write

$$f(x) = \frac{g(x)}{h(x)} = \frac{4x + 2}{x^2 + 3x + 2}.$$

In this case the denominator $h(x)$ has zeros at $x = -1$ and $x = -2$; Thus the partial fraction expansion will be of the form

$$f(x) = \frac{4x + 2}{x^2 + 3x + 2} = \frac{A_1}{x + 1} + \frac{A_2}{x + 2} \Rightarrow 4x + 2 = A_1(x + 2) + A_2(x + 1) \Rightarrow A_1 = -2 \qquad A_2 = 6.$$

Example 1.4: If the denominator has repeated factors, the expansion is carried out as follows:

$$f(x) = \frac{x - 4}{(x + 1)(x - 2)^2} = \frac{A}{x + 1} + \frac{Bx + C}{(x - 2)^2} \Rightarrow A = -\frac{5}{9} \qquad B = \frac{5}{9} \qquad C = -\frac{16}{9}.$$

1.6 Binomial expansion

We consider the general expansion of $f(x) = (x + y)^n$, where x and y may stand for constants, variables or functions and n is a positive integer.

$$\begin{aligned}(x + y)^1 &= x + y,\\ (x + y)^2 &= (x + y)(x + y) = x^2 + 2xy + y^2,\\ (x + y)^3 &= (x + y)(x^2 + 2xy + y^2) = x^3 + 3x^2y + 3xy^2 + y^3.\end{aligned}$$

The general expression, the binomial expansion for power n is given by

$$(x + y)^n = \sum_{m=0}^{n} C_{n,m} x^{n-m} y^m,$$

where

$$C_{n,m} \equiv \frac{n!}{m!(n - m)!} \equiv \binom{n}{m}.$$

1.7 The principle of mathematical induction (MI or mi)

Suppose $P(k)$ means that the property P holds for the integer k.
Suppose that m is an integer and that
(1) $P(m)$ is true,
(2) Whenever $k \geq m$ and $P(k)$ is true, $P(k + 1)$ is true.
Then $P(n)$ is true for all integers $n \geq m$.
(1) is called the **Base** and (2) is called the **Induction Step.** The supposition that $P(k)$ is true is called the **inductive hypothesis.**

Example: Sum of the first n integers.

$$1 + 2 + \cdots + n = \frac{n(n + 1)}{2} \qquad P(n)$$

Proof by MI:
Base: $P(1)$ is true as $1 = \frac{1(1+1)}{2}$.
Induction step: Let $k \geq 1$ and suppose $P(k)$ is true.
Thus $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$.
Hence $1 + 2 + \cdots + k + (k + 1) = \frac{k(k+1)}{2} + (k + 1) = \frac{(k+1)(k+1+1)}{2}$, and therefore $P(k + 1)$ is true as well.

Problem: Let $P(n)$ stand for "$6^{2n} - 1$" is divisible by 35.
Claim: $P(n)$ is true for all integers $n \geq 1$.
Proof by MI:
Base: $P(1)$ is true as $6^{2n} - 1 = 6^{2 \cdot 1} - 1 = 36 - 1 = 35$.

Induction step: Let $k \geq 1$ and suppose $P(k)$ is true. Thus $6^{2k} - 1$ is divisible by 35. Hence

$$6^{2(k+1)} - 1 = 6^{2k+2} - 1 = 6^{2k} \cdot 36 - 1 = 6^{2k} \cdot (35 + 1) - 1 = (6^{2k} - 1) + 6^{2k} \cdot 35,$$

which is divisible by 35, and therefore $P(k+1)$ is true as well.

Exercise: Let $P(n)$ stand for "$n^4 + 2n^3 + 2n^2 + n$" is divisible by 6.
Claim: $P(n)$ is true for all integers $n \geq 1$.
Proof by MI: `Do as an exercise at home.`

Theorem 1.2: If $x \neq 1$, then

$$\sum_{m=0}^{n} x^m = \frac{1 - x^{n+1}}{1 - x} \quad \text{for all integers} \quad n \geq 0.$$

Proof by MI: `Do as an exercise at home.`