MATH242001

This question paper consists of 3 printed pages, each of which is identified by the reference **MATH242001**

Only approved basic scientific calculators may be used.

Examination for the Module MATH2420
(January 2005)

MULTIPLE INTEGRALS AND VECTOR CALCULUS

Time allowed: **2 hours**

Attempt 4 out of 5 questions.

All questions carry equal marks.

1. (a) Consider the curves

$$C_1 : \underline{r}_1(t) = (t)\underline{i} + (t^2 - 6)\underline{j} + (t + 2)\underline{k}$$
$$C_2 : \underline{r}_2(u) = (u+1)\underline{i} + (u^2 - 1)\underline{j} + (u^3 - 3)\underline{k}.$$

Verify that they intersect at the point $P(3, 3, 5)$. At P determine the tangents to the curves and then find the cosine of the angle between the curves.

(b) Consider the function $f(x, y, z) = 2x^2 + 3xy + xy^2 - y^3 + z$.

(i) Find ∇f.

(ii) At the point $P(1, 2, 0)$, calculate the value of ∇f and hence give the maximum rate of increase of f and the direction (unit vector) of this steepest increase.

(iii) At P, find the rate of increase of f in the direction of the vector $(2, 4, 4)$.

(c) Let f be a function of x and y with $x = st^2 + s^2t$ and $y = st$.

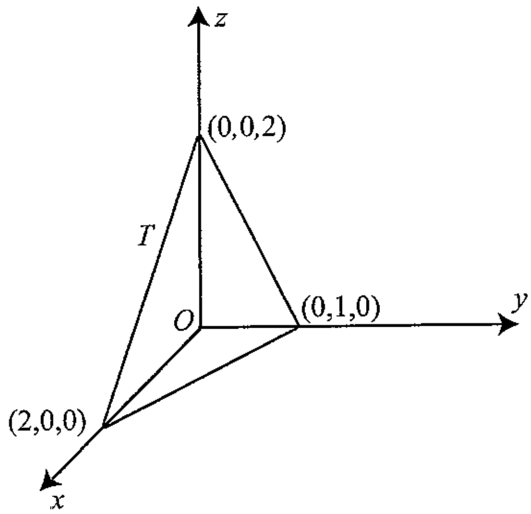
(i) Use the chain rule to express f_s and f_t in terms of f_x, f_y, s and t.

(ii) Then determine f_{st} in terms of f_{xx}, f_{xy}, f_{yy}, f_x, f_y, s and t.

2. (a) Integrate x over the region enclosed by the curves $y = 3x$ and $y = x^2 - 4$.

(b) Find the length of the curve $\underline{r}(t) = 3t^{3/2}\underline{i} + 2t\underline{j}$ from the point $(3, 2)$ to the point $(24, 8)$. You can leave your answer in the form $\frac{1}{a}[(b)^{3/2} - (c)^{3/2}]$, where a, b and c are integers.

(c) Use triple integration to find the volume of the tetrahedron T shown in the figure



[Hint: use the equation $ax + by + cz = 1$ to establish the equation of the plane passing through $(2, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 2)$.]

3. (a) (i) Calculate the Jacobian

$$\frac{\partial(u, v)}{\partial(x, y)}$$

for $u = x^2 + y^2$, $v = x^2 - y^2$.
Hence state the inverse Jacobian $\frac{\partial(x, y)}{\partial(u, v)}$.

(ii) Using the change of variables in (i) evaluate $\int\int_\Omega xy \, dx \, dy$, where Ω is the region in the first quadrant bounded by the curves $x^2 + y^2 = 1$, $x^2 + y^2 = 4$, $x^2 - y^2 = \frac{1}{4}$ and $x^2 - y^2 = 1$. [Hint: Substituting for $x = x(u, v)$ and $y = y(u, v)$ should **not** be necessary.]

(b) Integrate the vector function $\underline{h}(x, y, z) = (xy^2, yz^2, xz)$ over the curve C defined by $\underline{r}(t) = (t, t^2, t^2)$ from the point $P(1, 1, 1)$ to the point $Q(2, 4, 4)$, namely $\int_C \underline{h} \cdot d\underline{r}$.

(c) Consider the region in the first quadrant between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$ in the plane $z = 0$. Calculate the integral of $f(x, y) = x^2 + y^2 + x$ over this region.

4. (a) Let $\underline{u} = xyz^2\underline{i} + \cos(x^2yz)\underline{j} + \exp(xy^2z)\underline{k}$. Calculate

(i) $\nabla \cdot \underline{u}$,

(ii) $\nabla \times \underline{u}$.

(b) Consider the region of the xy-plane given by

$$(x^2 + y^2)^2 \leqslant (y^2 - x^2).$$

(i) Convert the region in plane polar coordinates.

(ii) Hence or otherwise, sketch the region.

(c) Green's theorem states that if $P(x,y)$ and $Q(x,y)$ are scalar functions defined over a domain Ω with a piecewise smooth boundary C, then

$$\int\int_{\Omega}\left[\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right]\mathrm{d}x\,\mathrm{d}y=\oint_C (P\,\mathrm{d}x+Q\,\mathrm{d}y)$$

where the line integral is taken anticlockwise around C.

(i) Explain how this is a special case of the Stokes's theorem for a smooth surface S with smooth bounding curve C and unit normal n, namely

$$\int\int_S (\nabla\times\underline{F})\cdot\underline{\hat{n}}\,\mathrm{d}S=\oint_C \underline{F}\cdot\mathrm{d}\underline{r}$$

(ii) If the region Ω is given by the rectangle $1\leqslant x\leqslant 2$, $0\leqslant y\leqslant 2$, calculate

$$\oint (xy^2)\,\mathrm{d}x+(2xy^2)\,\mathrm{d}y$$

both directly and using Green's theorem.

5. (a) The area of the surface S of a graph formed by $z=f(x,y)$ with $(x,y)\in\Omega$ is given by

$$\int\int_{\Omega}\left[\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1\right]^{1/2}\mathrm{d}y\,\mathrm{d}x.$$

Use this expression to evaluate the surface area of that part of the parabolic cylinder $z=x^2$ that lies over the triangle with vertices $(0,1)$, $(1,1)$ and $(0,0)$ in the xy-plane.

(b) A sphere of radius a centred at the origin can be parametrised by setting

$$\underline{r}(\theta,\phi)=(a\sin\theta\cos\phi,a\sin\theta\sin\phi,a\cos\theta),$$

where $(\theta,\phi)\in[0,\pi]\times[0,2\pi)$. Hence show that

$$\underline{N}=\underline{r}'_\theta\times\underline{r}'_\phi=a\sin\theta\underline{r}.$$

(c) Assuming that the flux of $\underline{q}$ across the above sphere in the direction of the unit normal $\underline{n}=\underline{r}/a$ to the sphere is

$$\int\int_S \underline{q}\cdot\underline{n}\,\mathrm{d}\theta=\int\int_{\Omega}\underline{q}(\underline{r}(\theta,\phi))\cdot\underline{n}|\underline{N}|\,\mathrm{d}\theta\,\mathrm{d}\phi=\int\int_{\Omega}\underline{q}(\underline{r}(\theta,\phi))\cdot\underline{N}\,\mathrm{d}\theta\,\mathrm{d}\phi$$

calculate the flux of the vector function $\underline{q}=(x,y,0)$ across the surface in the outward direction.

(d) Validate this results using the volume integral in the divergence theorem

$$\int\int_S \underline{q}\cdot\underline{n}\,\mathrm{d}\sigma=\int\int\int_V \nabla\cdot\underline{q}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

[Hint: First convert the volume integral into spherical polar coordinates.]

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